Abstract. In this paper we propose a refinement of dynamic equilibria based on small random perturbations of perfect foresight equilibrium paths in a class of one step forward looking dynamic models. Specifically, when the backward perfect foresight (bpf) map is a unimodal function exhibiting cycles or complex dynamics, we define a small random perturbation from the perfect foresight equilibrium as a sequence of random variables generated from small stochastic errors on the perfect prevision of the future state variable value. The stochastic process generated in this way is stationary provided that the support of the perturbation is small enough. Moreover, when the bpf map exhibits ergodic chaos, we show that the corresponding stationary measure converges to the Bowen-Ruelle-Sinai invariant measure of the bpf map as the size of the perturbation approaches zero. On the other hand, if the bpf map has an attracting cycle, then the stationary measure is close to a convex linear combination of Dirac measures supported on that cycle, converging to it as the size of the perturbation approaches zero. Therefore, for full measure sets of parameter values of a large class of one-parameter families of unimodal bpf maps, the stationary measures associated with the small random perturbation under consideration herein are either close to a combination of Dirac measures supported on a determinate cycle or to the absolutely continuous B-R-S measure associated with the deterministic (chaotic) bpf map or the chaotic sunspot equilibrium defined by Araujo and Maldonado (2000). Neither indeterminate cycles nor local sunspot equilibrium are found as the limiting behavior of the small random perturbation. Finally, we provide two examples -- the classical overlapping generations model with fiat money and the Shapley-Shubik market game - to illustrate the refinement of the dynamic equilibria in those models given by a small random perturbation of a perfect foresight equilibrium.

Keywords: Equilibrium selection; Sunspot equilibrium; Stochastic dynamical system; Stochastic stability.

JEL classification: C61, D84, E32

1. Introduction

The Rational Expectations Hypothesis (REH) requires not only individuals maximizers of their objective functions, but also consistency between the perceived randomness of future variables and their actual distribution. An even stronger concept is that of perfect foresight equilibrium, proposing that agents are able to have a perfect prevision of the exact value of the future state variable.

The strong version of the REH given by the perfect foresight assumption allows to find important equilibrium paths as indeterminate steady states or cycles and chaotic paths [10, 14, 20, 26]. From those equilibria, it is possible to construct stochastic equilibria based on extrinsics, the so called “sunspot equilibria” [1, 5, 16]. Although the perfect foresight assumption gives us a clear idea of the distinct types
of equilibria that a system may have, the question of what would happen if such assumption is not completely satisfied remains open; for instance, how the state variable sequence behaves under a small random perturbation of a perfect foresight equilibrium.

In this paper we provide a selection criterion of those dynamic equilibria for certain one-step forward looking economic models. Such selection criterion or refinement of dynamic equilibria is based on a small random perturbation of a perfect foresight equilibrium. The stochastic process generated in that way is stationary and, for a typical family of unimodal backward perfect foresight (bpf) maps, distinct outcomes may occur, depending on the parameter values of the model. If the unperturbed bpf dynamics exhibits an attracting cycle, the stationary measure associated with the small random perturbation is close to an atomic measure with support on that attracting cycle. More interestingly, in the case where the bpf dynamics are chaotic and an ergodic and absolutely continuous (with respect to Lebesgue) invariant measure for it exists, then the small random perturbation gives rise to a stochastic process with a stationary measure that is close to the absolutely continuous B-R-S measure of the bpf map, which is also the stationary measure of the global chaotic sunspot equilibrium presented in Araujo and Maldonado [1]. Additionally, as the maximal size of the random perturbation goes to zero, the stationary measures of the stochastic processes converge to the invariant measures associated with the bpf map. As a consequence, we assert that both the atomic measures with support on attracting cycles and the stationary measure of the chaotic sunspot equilibrium are stable under the refinement criterion of dynamic equilibria given by the small random perturbation of the perfect foresight equilibrium. It is also worth noting that an attracting cycle of the bpf map corresponds to a deterministic cycle in the forward perfect foresight dynamics. Therefore, neither indeterminate cycles (or steady states) nor other kind of sunspot equilibrium other than the global chaotic sunspot equilibrium may be selected under the refinement criterion proposed herein.

We can interpret the random perturbation as small expectations coordination failures (ECF) in a given model. The ECF have been a concern in the recent literature of dynamic models with heterogeneous agents [2, 18, 28, 29, 49], and since most of macro models deal with representative agents, this can be a simple way to introduce that imperfection into those models. Thus, within the setup of bpf maps belonging to an one-dimensional parameter family of unimodal maps with negative Schwarzian derivative and non-degenerate critical point, small failures in coordination will produce stochastic processes with stationary measures close to the one associate with a chaotic SE, in the case where the original unperturbed dynamics are chaotic on a finite union of intervals. When the deterministic dynamics exhibits an attracting cycle, the stationary measure of the perturbed system will remain close to an atomic measure with support on a periodic orbit. It is significant to remark that the two cases listed above correspond to a full measure set in parameter space.

The term sunspot was coined by Cass and Shell [15] who defined the concept of sunspot equilibrium in the context of general equilibrium to study the influence of the agents expectations on market outcomes. More recently, Lucas and Stokey [38] have argued that sunspots and contagion effects are sources of liquidity crises. They brought the argument of Cass and Shell [15] on expectation coordination to
explain the bank runs and consequently, the financial crises of 2008. Applications of the concept of sunspot equilibrium include the modeling of such bank-runs [43], lotteries [45, 46] and behavioral economics [24].

The existence of sunspot equilibrium requires that agents not only coordinate expectations, but also that they have to do it by using an extrinsic event. In spite of this seemingly strong hypothesis, our result asserts that allowing for a small random perturbation of the perfect foresight equilibrium, the resulting stochastic dynamics are almost the same.

In the literature we find other selection criteria of dynamic equilibria, most of them applied to linear or monotonic dynamic models of rational expectations. These selection criteria are important because typically those models exhibit multiplicity of equilibria, thus refinements of those equilibria are needed. The first criterion is the well-known local stability criterion [9, 11]. This simply proposes that the unique equilibrium must be stable under small perturbations of the initial condition. The second criterion is the minimal state variable criterion [40, 41] which states that the equilibrium selection must be done using forecasting functions with a minimal set of state variables and with parameters that are continuous functions around key values of the structural parameters. A recently defined criterion is the expectational stability [22, 23, 37], where the parameters defining the equilibrium must be stable under updating rules in meta-time. These dynamics represent the continuous and instantaneous revision of beliefs regarding those parameters. Finally, Driskill [21] provided a new selection criterion of equilibria, the “finite-horizon” or “backward-induction” criterion, consisting in taking the finite horizon model associated to the original one and finding the limit of the finite-horizon equilibrium as time goes to infinity. Our work complements the criteria already proposed in the literature, by focusing in the case of truly non-linear dynamics governing the state variable evolution.

The dynamics of unimodal maps, a key topic in the modern dynamical system theory (see e.g. [19] for an overview), plays a central role in our analysis. Among these maps, the ones with the simplest dynamical behavior are called hyperbolic: those unimodal maps with a hyperbolic attractor. A recent result of Kozlovski ensures that the set of hyperbolic unimodal maps is open and dense in the space of $C^r$ unimodal maps for any $r \geq 2$ [35, 36]. On the opposite side of the spectrum of dynamical complexity are those unimodal maps having an absolutely continuous invariant probability measure. Benedicks and Carleson [7] provide conditions satisfied by a positive measure set of values of the parameter of the quadratic family and implying the conclusion of Jakobson’s theorem [30], namely, that such maps admit a unique invariant probability measure which is absolutely continuous with respect to Lebesgue. See, e.g. [12, 32, 42], for more results regarding the existence of an absolutely continuous invariant measure for a unimodal map. A notion that will be of great relevance for the results herein, is the stability of dynamical systems under perturbations given by sequences of independent and identically distributed random variables, known as stochastic stability and initially introduced by Kolmogorov and Sinai [48]. Results in this direction have been obtained by Kifer [33, 34] for uniformly expanding maps, Axiom A attractors and geometric Lorenz attractors of flows, and Young [50] for the stochastic stability of Axiom A diffeomorphisms. In what concerns one-dimensional maps, Katok and Kifer [31] proved stochastic stability for the quadratic family in the Misiurewicz case. This result was later extended
for sets of values of the parameter with positive Lebesgue measure by Benedicks and Young [8], with respect to the convergence induced by the weak∗-topology, and by Baladi and Viana [6], with respect to the norm topology and for a wider class of unimodal maps. More recently, Ávila and Moreira [3, 4] proved that quadratic maps are stochastically stable for Lebesgue almost every parameter value, their results holding also for generic families of unimodal maps of the interval.

This paper is organized as follows. In Section 2, we delimitate the class of economic dynamic models that we are going to consider, define the concept of small random perturbation of a perfect foresight equilibrium and state our main theorem. In Section 3 we illustrate our main result through two examples. The first one is the classical overlapping generations model with fiat money and the second one is the Shapley-Shubik market game model. For a full measure set of values of the risk aversion parameter (for the first model) and of the market thickness parameter (for the second model), we find that the stationary measure describing the small random perturbation of the perfect foresight equilibrium is close to: i) an absolutely continuous B-R-S measure for the bpf map, corresponding to the stationary measure of the global chaotic sunspot equilibrium, whenever the backward perfect foresight map is chaotic on a finite union of intervals or ii) an atomic measure with support on a periodic orbit whenever the backward perfect foresight map possesses an attracting cycle.

2. Framework and main results

We will first introduce the setup under consideration in this paper, along with two novel notions, central to our reasoning here: a stationary process corresponding to a small random perturbation of a perfect foresight equilibrium path and a selection criterion based on such process. We will then prove that, with respect to such selection criteria, for a large number of unimodal bpf maps, the selected equilibrium is either a determinate cycle or a stationary probability measure consistent with the absolutely continuous empirical measure defined by the chaotic backward perfect foresight path or with the stationary measure of the chaotic sunspot equilibrium.

2.1. Framework. We will consider one-period forward looking models of the type

\[ F(x_t) = E_t[G(\tilde{x}_{t+1})], \]

where \( F \) and \( G \) are differentiable functions defined in open subsets of \( \mathbb{R} \), \( x_t \) is the value of the state variable of the model in period \( t \), \( \tilde{x}_{t+1} \) is the random variable representing the possible values of the state variable of the model in period \( t + 1 \) and \( E_t[\cdot] \) is the mathematical expectation operator conditioned on the information available up to time \( t \). As usual, the interpretation of the model (2.1) is the following: if the agents believe that the probability distribution of the state variable in period \( t + 1 \) is given by that of the random variable \( \tilde{x}_{t+1} \), then the current state variable value that equilibrates individual decisions and markets is \( x_t \).

When \( F \) is an invertible function it is possible to define the backward perfect foresight map, as follows.

Definition 2.1. The backward perfect foresight map (bpf map) associated to the model (2.1) is a real valued map such that \( F(\phi(x)) = G(x) \) for all \( x \) in the domain of \( \phi \).
Thus, if there exists $F^{-1}$, the bpf map is defined by $\phi(x) = F^{-1}(G(x))$. In general, $\phi(x)$ represents the current value of the state variable that equilibrates the markets when the expectation regarding the future value of the state variable is the Dirac measure $\delta_x$. That map allows us to define perfect foresight paths for the model.

**Definition 2.2.** A perfect foresight equilibrium path from $x_0$ is a sequence $\{x_t\}_{t \geq 0}$ such that $x_t = \phi(x_{t+1})$ for all $t \geq 0$.

Therefore, in a perfect foresight equilibrium path, individuals have perfect foresight regarding the future value of the state variable given by $x_{t+1}$, and the corresponding current value is $x_t = \phi(x_{t+1})$. The key point to take from this definition is that, even the future state value is known, the current state can be subject to some uncertainty. Thus, we introduce the following definition.

**Definition 2.3.** Let $\{\bar{\varepsilon}_t\}_{t \geq 0}$ be a sequence of independent and identically distributed random variables with density $\theta_\varepsilon$ with support $[-\varepsilon, \varepsilon]$. The associated small random perturbation of the perfect foresight equilibrium path from $x_0$ is a sequence $\{	ilde{x}_t\}_{t \geq 0}$ such that $\tilde{x}_0 = x_0$ and $\tilde{x}_t = \phi(\tilde{x}_{t+1}) + \bar{\varepsilon}_t$ for all $t \geq 0$.

The reasons to introduce the definition above are the following. First, we want to analyze the dynamics of the process governing the state variable under the possibility of a small random perturbation of the perfect prevision. Second, we want to find the asymptotic behavior associated these dynamics when the support of the small random perturbation shrinks to the singleton $\{0\}$. This is an important analysis that allows us to test the robustness of equilibrium paths under perfect foresight. Furthermore, it will provide a criterion for equilibrium selection when a multiplicity of equilibria is present. Third, as long as most of the macro models deal with a representative agent, this (small) random perturbation can be interpreted as representing expectations coordination failures in more general models with heterogeneous agents. Finally, we will see that the limiting dynamics is closely related to the concepts of determinate cycles and chaotic sunspot equilibrium.

The reading of Definition 2.3 should be done as follows. Given the actual state variable value for the next period, $\tilde{x}_{t+1}$, the current value for the state variable that rationalizes $\tilde{x}_{t+1}$ should be $\phi(\tilde{x}_{t+1})$. However, due to imprecisions in coordination of the future variable or in the exact specification of the model, a random perturbation of this value is observed; then the actual current state variable value that rationalizes $\tilde{x}_{t+1}$ is $\tilde{x}_t = \phi(\tilde{x}_{t+1}) + \bar{\varepsilon}_t$. According to this, the process $\{	ilde{x}_t\}_{t \geq 0}$ is a backward Markovian stochastic process, that is

$$P(\tilde{x}_t \in A | \tilde{x}_{t+1} = \alpha_1, \tilde{x}_{t+2} = \alpha_2, \ldots) = P(\tilde{x}_t \in A | \tilde{x}_{t+1} = \alpha_1)$$

for every Borelian $A \subset \mathbb{R}$ and every sequence of real numbers $\{\alpha_i\}_{i \geq 1}$.

Therefore, a small random perturbation of the perfect foresight equilibrium path is a sequence of random variables satisfying

$$\tilde{x}_t = \phi(\tilde{x}_{t+1}) + \bar{\varepsilon}_t .$$

Thus, given the expectation of the future state variable value $\tilde{x}_{t+1} = x$, and a random perturbation with support $[-\varepsilon, \varepsilon]$ as in Definition 2.3, the probability that the current value of the state variable belongs to the a Borel set $A \subset \mathbb{R}$ is given by:

$$P_\varepsilon(x, A) = \int_{A-\phi(x)} \theta_\varepsilon(t) \, dt = \int_A \theta_\varepsilon(y - \phi(x)) \, dy ,$$

where $	heta_\varepsilon$ is the density of the random variable $\bar{\varepsilon}$.
where \( A - \phi(x) \) denotes the subset of \( \mathbb{R} \) whose elements admit a representation of the form \( a - \phi(x) \), for some \( a \in A \).

If we suppose that the small random perturbation represent failures in expectations coordination, (2.3) represents the probability of the current value of the state variable being in \( A \) given that the expectation of the value for the future state is \( x \) and there is expectations coordination failures. Thus, \( P_\epsilon(x, A) \) can be seen as the backward conditional probability induced by the ECF process. Moreover, the Markov process with transition probability given by (2.3) has an absolutely continuous stationary measure, which we denote by \( \nu^\epsilon \) and satisfies the identity:

\[
\nu^\epsilon(A) = \int P_\epsilon(x, A) \, d\nu^\epsilon(x) .
\]

We are now in a position to introduce the central notion of this paper: an equilibrium selection criterion based on a small random perturbation.

**Definition 2.4.** An equilibrium of the one-period forward looking model (2.1) is selected by the criterion of the small random perturbation of the perfect foresight path equilibrium if its stationary measure is the limit of the empirical measures associated to the small random perturbation of perfect foresight equilibrium paths as the path length tends to infinity and the random perturbation maximum size tends to zero.

We should remark that the equilibrium that is eventually selected by the criteria stated above can, a priori, be of any of the following types: a steady state, a cycle (determinate or indeterminate), a chaotic backward perfect foresight path (indeed, its associated empirical measure) or a sunspot equilibrium (local or chaotic). Nonetheless, we will see that for a large number of certain nonlinear bpf maps only some of these will eventually occur: the selected equilibrium may be either a determinate cycle, a chaotic backward perfect foresight path or a chaotic sunspot equilibrium, but never an indeterminate cycle or any associated local sunspot equilibrium.

2.2. **The random perturbation selection criterion for generic families of unimodal bpf maps.** Let us denote by \( \mathcal{F} \) the class of \( C^3 \) unimodal bpf maps \( \phi \) defined on an interval of the form \([0, \alpha]\), with \( 0 < \alpha \leq +\infty \), and for which the following conditions hold:

1. \( \phi \) has a non-degenerate critical point \( x^* \) with \( \phi''(x^*) > 0 \);
2. \( \phi \) has a repelling fixed point at zero, i.e., \( \phi(0) = 0 \) and \( \phi'(0) > 1 \);
3. \( \phi \) has negative Schwarzian derivative.

Recall that the random perturbation introduced in Definition 2.3 is given by a sequence \( \{\epsilon_t\}_{t \geq 1} \) of independent and identically distributed random variables with density \( \theta_\epsilon \) with support \([-\epsilon, \epsilon]\). Additionally, we will assume from now on that some additional “smoothness conditions” are satisfied (Hypothesis B in Appendix A). Common densities such as, e.g., the truncated Normal density or the Uniform density, fulfill such technical conditions.

It is known that unimodal maps \( \phi \) with a non-degenerate critical point and negative Schwarzian derivative fit into three alternative topological types (see [19]):

1. The critical point \( x^* \) is such that \( \phi''(x^*) \neq 0 \).
2. The Schwarzian derivative is defined by \( S\phi = \frac{\phi'''(x)}{\phi''(x)} - \frac{3}{2} \left( \frac{\phi''(x)}{\phi'(x)} \right)^2 \). See the textbook [19] for a detailed treatment of one-dimensional dynamics.
The map $\phi$ has a periodic attractor whose basin is big both from a topological point of view (open and dense set) and in a measure-theoretical sense (full measure). Both the periodic attractor and its basin are stable under deterministic $C^1$ perturbations of $\phi$. Unimodal maps such as these are usually called hyperbolic or regular.

The map $\phi$ is transitive on some finite union of intervals, i.e. there exist orbits which are dense in these intervals, and $\phi$ has a B-R-S measure which is absolutely continuous with respect to Lebesgue, is supported on that finite union of intervals, and has a positive Lyapunov exponent. Even if $\phi$ may be unstable under deterministic perturbations (nearby maps may have a periodic attractor), the stochastic description given by the B-R-S measure is robust under stochastic perturbations in the sense that the perturbed system has a stationary measure whose density is close to B-R-S measure density with respect to the $L^1$ distance. This is case into which the bpf maps of subsection 2.3 fall.

The map $\phi$ is infinitely renormalizable and has a unique invariant probability measure $\mu$, which is a B-R-S measure supported in the closure of the forward orbit of the critical point of $\phi$. This set is an attracting Cantor set $C$ and if $x$ is in the basin of $C$ then

$$\lim_{n \to +\infty} \frac{1}{n} \sum_{i=0}^{n-1} f(\phi^i(x)) = \int f(x) \, d\mu$$

for every continuous function $f$.

A remarkable theorem by Avila and Moreira [3] asserts that for topologically generic $k$-parameter families of $C^r$, $r = 3, 4, \ldots, \infty$, families of unimodal maps with non-degenerate critical point and negative Schwartzian derivative, almost every non-hyperbolic parameter satisfies the subexponential recurrence condition and the Collet-Eckmann condition (see Appendix A for further details). As a consequence, within generic families of unimodal maps with non-degenerate critical point and negative Schwartzian derivative, almost all maps are stochastically stable (see the monograph [13] and references therein for further details). Moreover, as pointed out by Avila and Moreira, their result still holds if the negative Schwartzian derivative condition is removed. However, uniqueness of the B-R-S measure is no longer guaranteed. In what concerns the critical point non-degeneracy assumption, this is a typical condition among unimodal maps.

In what concerns the “relative sizes” of the three alternative topological types listed above, it should be remarked that the first two alternatives are observable for open sets of families of unimodal maps, with the first alternative also known to be dense within the class of unimodal maps under consideration here. Regarding the third alternative, this occurs only for a subset of Lebesgue measure zero of parameter space.

The next theorem relies heavily on Avila-Moreira Theorem [3] to provide a description of the eventual outcomes produced by random perturbation selection criterion of Definition 2.4. See Appendix B for a proof.

**Theorem 2.5.** For topologically generic $k$-parameter families of bpf maps in the class $\mathcal{F}$, the random perturbation selection criterion of Definition 2.4 will select a determinate cycle for an open and dense subset of the parameter space and, for
Lebesgue almost every other parameter, it will select equilibria associated with an absolutely continuous stationary measure.

It should be remarked that the random perturbation selection criterion of Definition 2.4 always picks a single equilibria for a very large class of unimodal maps – the hyperbolic ones. For such maps, the selected equilibria is always a determinate cycle. For almost every other value of the parameter (within generic parametric families), an equilibrium yielding an absolutely continuous stationary measure is selected. This can be one of the following, all consistent with the same absolutely continuous stationary measure $\mu$: a chaotic perfect foresight path whose empirical measure agrees with $\mu$ or a chaotic sunspot equilibrium with stationary measure $\mu$. We should also remark that indeterminate cycles or local sunspot equilibria are never selected by the proposed criterion. If the negative Schwartzian derivative condition is removed, then the absolutely continuous stationary measure $\mu$ is not necessarily unique. Nevertheless, even if a multitude of such measures is allowed to exist, then the equilibria selected by the random perturbation criterion proposed herein will always match one of these (finitely many) measures in the same sense as described above.

We recall that whenever the bpf map exhibits chaotic behavior, it has infinitely many (repelling) cycles. Therefore, model (2.1) exhibits the following distinct equilibria:

i) infinitely many indeterminate cycles;
ii) infinitely many local sunspot equilibria in the neighborhood of the indeterminate cycles;
iii) infinitely many chaotic perfect foresight equilibrium paths (realized by the bpf map $\phi$);
iv) exactly one chaotic sunspot equilibrium (with stationary measure $\mu$);

As noted earlier, out of all of the (infinitely many) equilibria listed above, only the last two emerge as possible outcomes of the random perturbation criterion selection whenever the original unperturbed bpf map exhibits chaotic dynamics.

In Section 3 we discuss two examples yielding an unimodal bpf map and we track the parameter values where the convergence of measure describing the small random perturbation process to either a B-R-S measure or to a determinate cycle is obtained.

2.3. The robustness of the chaotic sunspot equilibrium under random perturbation. Our next task is to prove that the stationary measures associated with the small random perturbation of the bpf map determined by (2.2) converge to the B-R-S measure $\mu$ of $\phi$, provided it exists, as the perturbation maximal size $\epsilon \rightarrow 0$. This will provide some robustness for the stationary measure of Araujo and Maldonado [1] sunspot equilibrium with respect to small random perturbation of the backward perfect foresight path.

Let $\epsilon_0$ be the largest positive real number such that $X_\epsilon \subset \mathbb{R}^+$ for every $\epsilon < \epsilon_0$. Given a partition $\mathcal{A} = (A_i)_{i=1}^n$ of $X_{\epsilon_0}$, define the map $P_{\mathcal{A}} : X_{\epsilon_0} \rightarrow \mathcal{A}$ by $P_{\mathcal{A}}(x) = A_i$ if and only if $x \in A_i$. Clearly, the map $P_{\mathcal{A}}$ is well defined for every $x \in X_{\epsilon_0}$. We define $\mu_N(P_{\mathcal{A}}(x))$ to be given by

$$\mu_N(P_{\mathcal{A}}(x)) = \frac{\#\{\phi^k(x) \in P_{\mathcal{A}}(x) : k \in \{0, \ldots, N-1\}\}}{N},$$
that is, the histogram defined by the deterministic $N$-history of the perfect foresight equilibrium path $\{x_t\}_{t \geq 0}$ and bin range defined by the partition $\mathcal{A}$. Denote the Lebesgue measure by $\lambda$ and the density of $\mu$ by $\rho$. Then, for $\mu$-a.e. $x \in X$ we have that

$$\rho_{\mathcal{A}}^N(x) := \frac{\mu_N(P_{\mathcal{A}}(x))}{\lambda(P_{\mathcal{A}}(x))}$$

is an estimate of the density (Radon-Nikodym derivative) of the invariant measure $\mu$ and, as $N \to \infty$ and the size of the partition $\mathcal{A}$ converges to zero, we get that

$$\rho_{\mathcal{A}}^N(x) \to \rho(x)$$

with respect to the $L^1$ norm.

Let us now consider a small random perturbation of a perfect foresight equilibrium path $\{\tilde{x}_t\}_{t \geq 0}$ associated with the stationary discrete-time process $\{\tilde{\epsilon}_t\}_{t \geq 1}$ described above, i.e. the terms of $\{\tilde{x}_t\}_{t \geq 0}$ satisfy the recursive relation (2.2). As discussed above, for every sufficiently small $\epsilon > 0$, the stochastic process defined by (2.2) has a unique ergodic absolutely continuous stationary measure $\nu^\epsilon$ with density $\rho^\epsilon$ (see [8]). Thus, an estimate of the measure $\nu^\epsilon$ can be defined by the asymptotic distribution of the Birkhoff average of Dirac measures over the path with expectations coordination failures $\{\tilde{x}_t\}_{t \geq 0}$ with $\tilde{x}_0 = x$, i.e. the measure

$$(2.5) \quad \nu^\epsilon_N := \frac{1}{N} \sum_{k=0}^{N-1} \delta_{\tilde{x}_{-k}}$$

converges to $\nu^\epsilon$ in the weak topology. Analogously to the deterministic case, for every $x \in X_\epsilon$ and every sufficiently small $\epsilon > 0$, $\nu^\epsilon_N(P_{\mathcal{A}}(x))$ is well defined by:

$$\nu^\epsilon_N(P_{\mathcal{A}}(x)) = \frac{\sharp \{ (\phi + \tilde{\epsilon}_k) \circ \cdots \circ (\phi + \tilde{\epsilon}_1)(x) \in P_{\mathcal{A}}(x) : k \in \{0, \ldots, N-1\} \} }{N}.$$

For $\nu^\epsilon$-a.e. $x \in X_\epsilon$, we can estimate the density $\rho^\epsilon$ of the measure $\nu^\epsilon$ by:

$$\rho^\epsilon_{\mathcal{A}}^N(x) := \frac{\nu^\epsilon_N(P_{\mathcal{A}}(x))}{\lambda(P_{\mathcal{A}}(x))}.$$ 

Moreover, we note that $\rho^\epsilon_{\mathcal{A}}^N(x) \to \rho^\epsilon(x)$ with respect to the $L^1$ norm as $N \to \infty$ and the size of the partition $\mathcal{A}$ converges to zero.

The next statement concerns the stochastic stability of backward perfect foresight maps. It provides conditions under which the stationary measure associated with the small random perturbation of the perfect foresight path converges to the B-R-S measure $\mu$ as the size of the random perturbation decreases to zero. For the family of backward perfect foresight maps under consideration here, this result follows as a consequence of Baladi–Viana Theorem regarding the strong stochastic stability of a certain family of unimodal maps [6]. See Appendix B for a proof.

**Proposition 2.6.** Assume that model (2.1) has a bpf map $\phi \in \mathcal{F}$, the random perturbation $\{\tilde{\epsilon}_t\}_{t \geq 1}$ is as described in Definition 2.3 and that some additional technical and verifiable conditions are fulfilled (Hypothesis A and B given in the Appendix A). Then, the following statements hold:

i) The approximate measure $\nu^\epsilon_N$ associated with the small random perturbation of a perfect foresight equilibrium path converges to the measure $\mu$ of the deterministic dynamics $\phi$ in the weak$^*$-topology, when $N \to \infty$ and $\epsilon \to 0$. 

ii) The approximate density $\rho_N^{A_\epsilon}$ associated with the small random perturbation of a perfect foresight equilibrium path converges to the density $\rho$ of $\phi$ in the norm topology, when $N \to \infty$, $\epsilon \to 0$ and for every sequence of partitions $A = (A_i)_{i=1}^\infty$ such that $\max_i |A_i| \to 0$. Proposition 2.6 has the following important consequences. Under suitable conditions, a sufficiently small random perturbation of a perfect foresight paths is stable in a stochastic sense, i.e. the stationary measures associated with the randomly perturbed processes are close to the invariant measures associated with the unperturbed path. Notice that we are not analyzing the perturbation of each element of the path $\{x_t\}_{t \geq 0}$, where $x_t = \phi(x_{t+1})$, but rather the stability (with respect to the random perturbation) of the whole system seen as a whole entity. Moreover, it should be noted that the benchmark measure against which the stability of the process is studied is the invariant B-R-S measure of the backward perfect foresight map $\phi$. Recalling Araujo and Maldonado [1] result, when the bpf map is a unimodal with an absolutely continuous invariant measure, there exists a stationary sunspot equilibrium for the model (2.1) given by the Markovian process

$$Q(x, A) = \frac{d\mu \circ f}{d\mu}(x)\delta_{f(x)}(A) + \frac{d\mu \circ g}{d\mu}(x)\delta_{g(x)}(A),$$

where $f$ and $g$ are the local inverses of $\phi$.

Remark 2.7. To define the sunspot equilibrium (2.6) Araujo and Maldonado [1] used the definition provided by [17]: A sunspot equilibrium for the model (2.1) is a subset $X_0 \subset X$ and a transition function $Q : X_0 \times B(X_0) \to [0, 1]$ such that

i) $F(x) = E_Q(x, \cdot)[G(\tilde{x})]$ for all $x \in X_0$; and

ii) there exists $x_0 \in X_0$ such that $Q(x_0, \cdot)$ is truly stochastic.

The sunspot equilibrium is stationary if there exists a stationary probability measure $\mu$ for the transition function $Q$. The transition function defined by (2.6) satisfies that definition above and the ergodic and $\phi$-invariant absolutely continuous measure $\mu$ is also invariant for that $Q$. It is clear that once a stationary sunspot equilibrium is found for (2.1) in that way, it can be defined the process $s_{t+1} = F(x_t) - G(x_{t+1})$, which explicitly represents the (extrinsic) sunspot variable.

Indeed, the stationary measure associated with the Markov process with transition function given by (2.6) is the B-R-S measure $\mu$. Thus, the stationary measure associated with the small random perturbation of the perfect foresight map converges to the stationary measure of the sunspot equilibrium of Araujo and Maldonado [1] when the maximal perturbation size converges to zero. This is a robustness result for that type of sunspot equilibria that fits the selection criteria introduced in Definition 2.4.

2.4. A brief discussion of a couple of alternative robustness criteria. In this subsection we very briefly discuss a couple of alternative formulations for the robustness of equilibria of model (2.1), modeling slightly different points of view.

The first such formulation resembles the small random perturbation approach introduced before, but relies on a different type of random perturbation. Acknowledging that coordination failures by the representative economic agents of model (2.1) may diminish as time advances, a phenomena that may eventually be related with some form of learning on the part of such agents, one could think of replacing the random perturbation of Definition 2.3, that is, a sequence of independent
and identically distributed random variables with fixed support, by a sequence of independent random variables \( \{\tilde{\eta}_t\}_{t \geq 1} \) with support \([-\tilde{\eta}_t, \tilde{\eta}_t]\), where \( \tilde{\eta}_t \) is a positive non-increasing function such that \( \lim_{t \to +\infty} \tilde{\eta}_t = 0 \). The shrinking support of the random variables \( \tilde{\eta}_t \) would then account for the diminishing coordination failures, while the rate of decrease of \( \tilde{\eta}_t \) would provide an indication towards how fast agents acquire information and readjust their expectations towards future events. We expect the analogues of the results stated in the previous sections to hold. Indeed, due to the shrinking support of the random perturbation, the rates of convergence to the limiting stationary measures should even be faster for this alternative setup.

The second formulation avoids the concept of random perturbation altogether. Instead, we look into perturbations of sunspots (either classical or chaotic). For ease of exposition, let us restrict our attention to the chaotic sunspots with transition functions of the form (2.6). Fix some transition function \( Q \) and consider perturbations \( Q_{\epsilon} \) which are sufficiently close to \( Q \) with respect to some appropriate norm. Then, it is unlikely that the perturbed transition function \( Q_{\epsilon} \) is still associated with an equilibrium for the original economy. Nevertheless, \( Q_{\epsilon} \) may eventually be realized as the transition function associated with a chaotic sunspot equilibrium of a nearby economy, determined by functions \( F_{\epsilon} \) and \( G_{\epsilon} \) sufficiently close to the functions \( F \) and \( G \) of (2.1), respectively. However, the construction of the perturbations \( F_{\epsilon} \) and \( G_{\epsilon} \) may be delicate. One way to work around this issue would be to perturb the functions \( F \) and \( G \) of model (2.1) within a class of pairs of functions that still yield unimodal bpf maps \( \phi_{\epsilon} \) (satisfying appropriate regularity conditions). Then, under suitable conditions, the bpf maps \( \phi_{\epsilon} \) associated with the perturbed model would still be close to the bpf map \( \phi \) determined by model (2.1). Moreover, the bpf map \( \phi_{\epsilon} \) could eventually determine a chaotic sunspot equilibrium of the form (2.6), close to the one determined by \( \phi \). A weakness of this approach may lie on the following fact: unimodal maps \( \phi \) yielding chaotic sunspot equilibria are known to be unstable under deterministic perturbations, even if stable under stochastic perturbations. Thus, there may be examples of perturbed bpf maps \( \phi_{\epsilon} \) arbitrarily close to \( \phi \), but without any chaotic sunspot equilibrium.

3. Two Examples

In this section we provide two examples where the convergence of the stationary measure associated with the small random perturbation of perfect foresight paths is illustrated. Depending on the parameter values of these models, as the maximal size of the random perturbation decreases to zero, we have convergence of the stationary measure to the absolutely continuous stationary B-R-S measure (associate also with the chaotic sunspot equilibrium (2.6)), provided the perfect foresight map is chaotic on a finite union of intervals, and convergence to a convex linear combination of Dirac measures supported on a periodic orbit, provided the perfect foresight map has an attracting cycle (namely, a locally deterministic cycle). Taking that convergence as a selection criterion when multiplicity of equilibria is present, we will have that the only equilibria that are selected by the criterion of small random perturbations of backward perfect foresight paths are either determinate cycles or a stationary probability consistent with the absolutely continuous empirical measure associated to a chaotic backward perfect foresight path or with the stationary measure of the chaotic sunspot equilibrium (2.6). Neither indeterminate equilibria nor
other types of classical local sunspot equilibria are selected by the small random perturbation criterion proposed herein.

The first example is the classical overlapping generation model with fiat money and the second is the market game model of Shapley and Shubik.

3.1. An OLG model with fiat money. In this section we will consider a two-period overlapping generations (OLG) model like the one introduced in [27] and provide an application for our main results. An analogous model was analyzed by Azariadis and Guesnerie in [5] to prove the existence of cycles and sunspots with finite support. We will show that depending on the parameter values of the model and up to a set of measure zero in parameter space, small perturbation from the perfect foresight equilibrium can be either close to the absolutely continuous B-R-S measure of the bpf map (and the stationary measure of the sunspot equilibrium \(2.6\)) or close to an atomic measure with support on an attracting cycle for the deterministic dynamics.

The economy is populated by a large number of young and old agents. The population sizes are the same and remain constant over time. We assume that there exists a representative agent with preferences given by a separable utility function

\[
U(c_t, c_{t+1}) = \frac{c_1^{1-\alpha_1}}{1-\alpha_1} + \frac{c_2^{1-\alpha_2}}{1-\alpha_2},
\]

where \(\alpha_1 > 0\) and \(\alpha_2 > 0\) are the coefficients of relative risk aversion of the agent, \(c_t, c_{t+1}\) denote the corresponding consumption plan. We suppose also that one unit of the good is produced with one unit of the unique productive factor (labor) and let \(l_1^*\) and \(l_2^*\) denote the agents labor endowments in the first and second periods of their lives, respectively. Finally, we assume that there is a risk-free asset (fiat money) that can be purchased by the agents providing a gross return \(z_t = 1\), and the money supply is constant, i.e. \(M_t = M_0\) for all \(t \geq 0\).

We will now state the consumption-saving problem of the representative agent. Let \(p_t\) and \(p_{t+1}\) denote the prices of the unique good in the economy during the first and second periods of the individual’s life. Note that while \(p_t\) is known by the individual during the first stage of her life, her knowledge of \(p_{t+1}\) consists of a probability distribution \(\mu_{t+1}\) representing the likelihood of occurrence of particular values of \(p_{t+1}\) and reflecting the individual’s beliefs about the state of economy during the second period of her life. The agent must choose a consumption plan \((c_t, c_{t+1})\) and the first period saving \(m_t\) as the solution of the following optimization problem

\[
(3.1) \quad \max_{\{c_t, c_{t+1}, m_t\}} \frac{c_1^{1-\alpha_1}}{1-\alpha_1} + E_t \left[ \frac{c_2^{1-\alpha_2}}{1-\alpha_2} \right]
\]

subject to the budget constraints

\[
\begin{align*}
p_t c_t + m_t &= p_t l_1^* \\
p_{t+1} c_{t+1} + m_{t+1} &= p_{t+1} l_2^* + m_t,
\end{align*}
\]

where \(E_t[\cdot]\) is the mathematical expectation with respect to the probability measure \(\mu_{t+1}\). Working out the first order condition for an interior solution of (3.1) leads to

\[
(3.2) \quad -\frac{1}{p_t} \left( l_1^* - \frac{m_t}{p_t} \right)^{-\alpha_1} + E_t \left[ \frac{1}{p_{t+1}} \left( l_2^* + \frac{m_t}{p_{t+1}} \right)^{-\alpha_2} \right] = 0 .
\]
The normalized monetary equilibrium condition is $M_t = m = 1$. After introducing the new variable $x_t = 1/p_t$, the first order condition (3.2) may be rewritten as

(3.3) \[ F(x_t) = E_t[G(x_{t+1})] \]

where $F$ and $G$ are auxiliary functions defined by

(3.4) \[ F(x) = x(l^*_1 - x)^{-\alpha_1} \quad \text{and} \quad G(x) = x(l^*_2 + x)^{-\alpha_2}, \]

thus yielding an identity of form (2.1). defining the equilibrium dynamics for the state variable $x_t = 1/p_t$.

The map $F: [0, l^*_1) \rightarrow \mathbb{R}^+$ defined in (3.4) is strictly increasing, and thus invertible. Hence, we obtain that the corresponding backward perfect foresight map is of the form

(3.5) \[ \phi(x) = F^{-1}(G(x)) \]

with the associate bpf dynamical system being determined by

\[ x_t = \phi(x_{t+1}) = F^{-1}(G(x_{t+1})) \]

3.2. The Shapley-Shubik market game model. We will now consider the one commodity overlapping generations version of the Shapley-Shubik market game [47], presented by Goenka et. al. [25].

This is a model of a pure exchange economy with overlapping generations of agents and a single good being traded. Time is discrete and labeled as $t = 1, 2, \ldots$. In each period $t > 0$ agents are born and live two periods. Thus, young and old individuals coexist in each period $t$, with $n$ elder individuals alive at time $t = 1$.

Individuals are assumed to have identical utilities for consumption and identical initial endowments in each period of life. Denoting by $(c^t_t, c^{t+1}_t)$ the consumption plan of an individual born in period $t$, her utility is given by:

(3.6) \[ U(c^t_t, c^{t+1}_t) = u_1(c^t_t) + u_2(c^{t+1}_t). \]

We will suppose that $U$ is a Von Newman-Morgenstern utility function; thus, in the presence of uncertainty, an expected value operator $E[\cdot]$ is set in the second term of (3.6). The initial endowments in each period of life are $\omega_1$ and $\omega_2$.

The trade mechanism is as follows. There exists a fixed amount $\bar{m}$ of fiat money in the economy. Each individual $i \in \{1, 2, \ldots, n\}$ born at time $t$ provides a (monetary) bid $b_{it} = (b^t_{it}, b^{t+1}_{it})$ to get monetary resources to buy goods, decides her savings $m_t$ and also provides an offer of goods $q_{it} = (q^t_{it}, q^{t+1}_{it})$. Bids and offers operations are performed in trading posts and in case of presence of uncertainty the negotiations are conditioned to each state of the nature. The aggregate bids and offers of individuals born at time $s = t, t + 1$ are defined by

(3.7) \[ B^s_t = \sum_{i=1}^n b^s_{it} \quad \text{and} \quad Q^s_t = \sum_{i=1}^n q^s_{it}, \quad s = t, t + 1 \]

and the bids and offers in the market are:

(3.8) \[ B_t = B^t_{t-1} + B^t_t \quad \text{and} \quad Q_t = Q^t_{t-1} + Q^t_t. \]

The terms of trade, that will be used as a “price” of the commodities in terms of fiat money, is defined by $B_t/Q_t$. Then, the lifetime budget constraints of each
individual are given by
\[(3.9) \quad b_t^t + m_t = \frac{B_t}{Q_t} q_t^t \quad \text{and} \quad b_{t+1}^t = m_t + \frac{B_{t+1}}{Q_{t+1}} q_{t+1}^t.\]

If (3.9) is satisfied, then the lifetime consumption of each individual is:
\[(3.10) \quad c_t^t = \omega_1 - q_t^t + \frac{Q_t}{B_t} b_t^t \quad \text{and} \quad c_{t+1}^t = \omega_2 - q_{t+1}^t + \frac{Q_{t+1}}{B_{t+1}} b_{t+1}^t.\]

To solve the individual problem in terms of the demand of money, we have to express the terms in (3.10) as functions of \(m_t\). To do this, let us express the aggregate bids (offers) in \(t\) and \(t+1\) in terms of the individual bid (offer) and the other agents aggregate bids (\(\hat{B}_t, \tilde{B}_{t+1}\)) (offers (\(\hat{Q}_t, \tilde{Q}_{t+1}\)) in \(t\) and \(t+1\) respectively:
\[(3.11) \quad B_t = b_t^t + \hat{B}_t \quad \text{and} \quad Q_t = q_t^t + \hat{Q}_t.\]

Analogously, we have that
\[(3.12) \quad B_{t+1} = b_{t+1}^t + \tilde{B}_{t+1} \quad \text{and} \quad Q_{t+1} = q_{t+1}^t + \tilde{Q}_{t+1}.\]

Substituting (3.11) into the first constraint in (3.9), we obtain the identity
\[(\hat{Q}_t + q_t^t)(b_t^t + m_t) = (\hat{B}_t + b_t^t)q_t^t,\]
which, rearranging terms, yields
\[(3.13) \quad b_t^t + m_t = \left[ \frac{\hat{B}_t - m_t}{Q_t} \right] q_t^t.\]

Similarly, substituting (3.12) into the second constraint in (3.9) results in:
\[(3.14) \quad b_{t+1}^t = m_t + \left[ \frac{\tilde{B}_{t+1} + m_t}{Q_{t+1}} \right] q_{t+1}^t.\]

Substituting (3.9) into (3.13) and (3.14), we obtain the terms of trade in each period, given by
\[(3.15) \quad \frac{B_t}{Q_t} = \frac{\hat{B}_t - m_t}{Q_t} \quad \text{and} \quad \frac{B_{t+1}}{Q_{t+1}} = \frac{\tilde{B}_{t+1} + m_t}{Q_{t+1}}.\]

Finally, putting together (3.9), (3.10), (3.13) and (3.15) (as well as (3.9), (3.10), (3.14) and (3.15)), we obtain the individuals lifetime consumptions in terms of the money demand and the strategies of all other individuals:
\[(3.16) \quad c_t^t(m_t) = \omega_1 - \frac{\hat{Q}_t}{\hat{B}_t - m_t} m_t \quad \text{and} \quad c_{t+1}^t(m_t) = \omega_2 + \frac{\tilde{Q}_{t+1}}{\tilde{B}_{t+1} + m_t} m_t.\]

If in period \(t\), agents do not know neither the value of the aggregate bid \(\hat{B}_{t+1}\) nor the value of the aggregate offer \(\hat{Q}_{t+1}\) for the next period, then they will use beliefs regarding these variables (\(\mu_{t+1}\)). Thus a mathematical expectation operator must have to be included for the utility of the second period of life. Therefore, the individual problem is to find the demand of money such that:
\[(3.17) \quad \max_{m_t} u_1(c_t^t(m_t)) + E_t \left[ u_2(c_{t+1}^t(m_t)) \right],\]
where $E_t[\cdot]$ is the mathematical expectation with respect to the probability measure $\mu_{t+1}$. The First Order Condition for interior solutions of the optimization problem above is:

\begin{equation}
(3.18) \quad \left( \frac{\hat{Q}_t}{\hat{B}_t - m_t} + \frac{\hat{Q}_tm_t}{(\hat{B}_t - m_t)^2} \right) u'_1 \left( \omega_1 - \frac{\hat{Q}_tm_t}{\hat{B}_t - m_t} \right) = \frac{\hat{Q}_t}{\hat{B}_t - m_t} - m_{t-1}
\end{equation}

and

\begin{equation}
\left[ \left( \frac{\hat{Q}_{t+1}}{\hat{B}_{t+1} - m_{t+1}} - \frac{\hat{Q}_{t+1}m_{t+1}}{(\hat{B}_{t+1} - m_{t+1})^2} \right) u'_2 \left( \omega_2 + \frac{\hat{Q}_{t+1}m_{t+1}}{\hat{B}_{t+1} - m_{t+1}} \right) \right] E_t
\end{equation}

It should be recalled that in this kind of models there is indeterminacy in either the offers or the bids (Peck et al. [44]). Thus, we will suppose that the sequence of offers $q_t, q_{t+1}$ is given and the amount of fiat money $\bar{m}$ is fixed. In this context we give the following definition of equilibrium.

**Definition 3.1.** Given the exogenous offers $(q_t, q_{t+1})$ and the amount of fiat money $\bar{m}$, a *monetary Nash equilibrium* is a sequence of bids $(\hat{b}_t, \hat{b}_{t+1})$ that makes $m_t = \bar{m}$ the solution of (3.18).

To obtain the dynamics of the bids, let us notice that from (3.15) one can get

\[ \frac{\hat{Q}_t}{\hat{B}_t + m_{t-1}} = \frac{\hat{Q}_t}{\hat{B}_t - m_t}, \]

which, rearranging terms, yields

\[ \hat{B}_t = \frac{\hat{Q}_t}{\hat{Q}_t} (\hat{B}_t - m_t) - m_{t-1}. \]

Substituting the identity immediately above into the right-hand side of (3.18) results in:

\begin{equation}
(3.19) \quad \left( \frac{\hat{Q}_t}{\hat{B}_t - m_t} + \frac{\hat{Q}_tm_t}{(\hat{B}_t - m_t)^2} \right) u'_1 \left( \omega_1 - \frac{\hat{Q}_tm_t}{\hat{B}_t - m_t} \right) = \frac{\hat{Q}_t}{\hat{B}_t - m_t} - m_{t-1}
\end{equation}

Suppose now that the sequence of exogenous offers is $q_t = q_1$ and $q_{t+1} = q_2$ for all $t \geq 1$ and $\bar{m} = m$. Then, for all $t \geq 1$, one must have that

\[ \hat{Q}_t = \hat{Q} = (n - 1)q_1 + nq_2 \quad \text{and} \quad \hat{Q}_t = \hat{Q} = nq_1 + (n - 1)q_2. \]

Substituting the two identities above into (3.19) yields the dynamics in the aggregate bids:

\begin{equation}
(3.20) \quad \left( \frac{\hat{Q}}{\hat{B}_t - m} + \frac{\hat{Q}m}{(\hat{B}_t - m)^2} \right) u'_1 \left( \omega_1 - \frac{\hat{Q}m}{\hat{B}_t - m} \right) = \frac{\hat{Q}}{\hat{B}_t - m} - m_{t-1}
\end{equation}

in this case, the mathematical expectation operator is with respect to the probability distribution of the next period aggregate bids $\hat{B}_{t+1}$. Finally, we introduce the
variable
\[ x_t = \frac{\hat{Q}m}{B_t - m} \]

and rewrite the equilibrium dynamics condition in (3.20) as
\[ \frac{(x_t^2 + \hat{Q}x_t)}{Q} u_1'(\omega_1 - x_t) = E_t \left[ \frac{(\hat{Q}x_{t+1} - x_{t+1}^2)}{Q} u_2'(\omega_2 + x_{t+1}) \right] , \]

where, without loss of generality, the induced probability distribution on \( x_{t+1} \) is also denoted by \( \mu_{t+1} \).

Let us now consider, as done in the example of Section 3.1, the case of constant relative risk aversion utility functions:
\[ u_i(c) = c^{1-\alpha_i} \]

or the limit case of logarithmic utility functions (when \( \alpha_i \to 1 \)):
\[ u_i(c) = \ln(c) . \]

Under these functional specifications, we are able to define
\[ F(x) := \frac{(x^2 + \hat{Q}x)}{Q} (\omega_1 - x)^{-\alpha_1} , \]
and
\[ G(x) := \frac{(\hat{Q}x - x^2)}{Q} (\omega_2 + x)^{-\alpha_2} , \]
yielding, in the general stochastic case, the dynamical system
\[ F(x_t) = E_t [G(x_{t+1})] , \]
which is again of the form (2.1).

Noticing that the function \( F \) is strictly increasing in \([0, \omega_1]\) and the function \( G \) is a unimodal function on the interval \([0, \hat{Q}]\), we obtain that the backward perfect foresight map
\[ \phi(x) = F^{-1}(G(x)) \]
is again a unimodal map. Moreover, as with (3.5), this map determines a dynamical system through the recursive relation.
\[ x_t = \phi(x_{t+1}) = F^{-1} (G(x_{t+1})) . \]

Due to the striking similarities between the the bpf maps (and corresponding dynamical systems) of the models in this section and the previous one, we move on to perform a combined analysis for these two cases.

3.3. Numerical simulations. In this last subsection we provide the outcomes of some numerical simulations of the two models discussed above. Our aim is to illustrate the convergence of the stationary measures associated with the small random perturbation of the perfect foresight equilibrium as the maximal size of the random perturbation tends to zero. Depending on the models’ parameter values, the limiting behavior of the corresponding stationary measure may be an absolutely continuous and ergodic invariant measure, in the case of deterministic dynamics which are chaotic in a finite union of intervals, and a linear convex combination of Dirac measures with support on an attracting cycle for the deterministic dynamics, for the case of regular deterministic bpf dynamics.
To keep things unified, let $\phi_{\lambda}(x)$ denote the bpf in any of the models described, where $\lambda$ is one of the parameters defining the function, taking values in a certain set $\Lambda$, as we describe below:

a) For the OLG model with fiat money of Section 3.1, we fix every parameter ($\alpha_1$ is fixed in such a way that $\alpha_1 \in (0,1]$), except for $\lambda = \alpha_2$, which is allowed to vary in the set

$$\Lambda = \{ \lambda \in [2, +\infty) : (l^*_1)^{\alpha_1} > (l^*_2)^{\lambda} \} .$$

Then, for every $\lambda \in \Lambda$, we have that $\phi_{\lambda}$ is a $C^3$ unimodal map with $\phi(0) = 0$, $\phi'(0) > 1$, a (positive) non-degenerate critical point $\bar{x}(\lambda)$ and negative Schwarzian derivative

$$S_{\phi}(x) < 0.$$

b) For the market game model of Section 3.2, we fix every parameter value associated with the relative risk aversion coefficients and initial endowments in such a way that $\omega_{\alpha_1} < \omega_{\alpha_2}$. The parameter $\lambda = \tilde{Q}$ is allowed to vary in the set

$$\Lambda = (0, +\infty) .$$

Under these conditions, for every $\lambda \in \Lambda$, $\phi_{\lambda}$ is a $C^3$ unimodal map with $\phi(0) = 0$, $\phi'(0) > 1$, a (positive) non-degenerate critical point $\bar{x}(\lambda)$ and $S_{\phi}(x) < 0$.

Henceforth, we will focus our attention on those two one-parameter families $\phi_{\lambda}$, $\lambda \in \Lambda$, of bpf maps of the examples above (parameterized by the relative risk aversion coefficient $\lambda = \alpha_2$ for the first example and the thickness parameter $\tilde{Q}$ for the second).

Before proceeding with our discussion, we need to introduce an auxiliary concept. Let $\Lambda$ be some interval in $\mathbb{R}$ and denote by $F_{\lambda}$ a one-parameter family bpf maps $\phi_{\lambda} \in F$, depending on a real parameter $\lambda \in \Lambda$, and for which the map $(x, \lambda) \mapsto (\phi_{\lambda}(x), D_x\phi_{\lambda}(x), D_x^2\phi_{\lambda}(x))$ is $C^1$. Denote by $\bar{x}(\lambda_*)$ the critical point of the unimodal map $\phi_{\lambda}$.

**Definition 3.2.** We say that the one-parameter family $\phi_{\lambda}$, $\lambda \in \Lambda$, has a Misiurewicz parameter $\lambda_* \in \Lambda$ with generic unfolding if the following conditions hold

a) $\lambda_* \in \Lambda$ is such that $\phi_{\lambda_*}$ is a Misiurewicz map, i.e. $\phi_{\lambda_*}$ has no periodic attractors and the forward critical orbit does not accumulate on its critical point;

b) the following transversality condition holds:

$$\lim_{n \to +\infty} \frac{D_{\lambda}\phi_{\lambda_*}^n(\bar{x}(\lambda_*))}{D_x\phi_{\lambda_*}^{n-1}(\phi_{\lambda_*}(\bar{x}(\lambda_*)))} \neq 0 .$$

In what concerns conditions in the definition above, we remark that in the case where $\phi_{\lambda_*}$ is a post-critically finite Misiurewicz map, i.e. $\phi_{\lambda_*}$ has no periodic attractors and some iterate $N$ of the critical point $\bar{x}(\lambda_*)$ reaches a repelling periodic point $P(\lambda_*)$, then condition b) is equivalent to the transversality of the curves $\lambda \mapsto \phi_{\lambda_*}^{N}(\bar{x}(\lambda_*))$ and $\lambda \mapsto P(\lambda)$.

The next result provides a set of conditions under which there exists a large set of bpf maps for which the strong stochastic stability of Proposition 2.6 holds.

---

+A negative Schwarzian derivative for a unimodal map guarantees that $\phi$ is topologically mixing in the interval where the dynamics takes place.
Nevertheless, we should remark that there may be additional parameter values yielding such strong stochastic stability. See Appendix B for a proof of the result.

**Theorem 3.3.** Let $\phi_\lambda, \lambda \in \Lambda \subseteq \mathbb{R}$, be a one-parameter family of bpf maps defined for one of the models above (the OLG model with fiat money of Section 3.1 or the market game model of Section 3.2).

(i) If $\phi_\lambda$ has a Misiurewicz parameter $\lambda_* \in \Lambda$ with generic unfolding, then there exists a positive measure set $A \subseteq \Lambda$ having $\lambda_*$ as a density point such that for every $\lambda \in A$ there exists an invariant measure $\mu_\lambda$ which is an absolutely continuous B-R-S measure.

(ii) Moreover, if the dynamical system determined by $\phi_\lambda$ is perturbed by a random process $\{\tilde{\epsilon}_t\}_{t \geq 1}$ such as described in Definition 2.3 (fulfilling certain additional technical and verifiable conditions: Hypothesis B on Appendix A), then the results of Proposition 2.6 are also valid.

As the theorems above asserts, the set of parameter values where the stochastic stability is valid has full Lebesgue measure for a generic parametric family of unimodal bpf maps in $\mathcal{F}$. In that set, the limit of the stationary measure of the small random perturbation of $\phi_\lambda$ when the size of the perturbation goes to zero, is either the absolutely continuous B-R-S measure of the bpf map (and the stationary measure of the chaotic sunspot equilibrium (2.6)) or a convex linear combination of Dirac measures supported on a determinate (locally unique) cycle of the model. Notice that repelling cycles (namely, indeterminate cycles of the model) are not found as the limiting behavior of the random perturbation. Since there exists a close relationship between indeterminate cycles (including steady states) and sunspots equilibria around them [5, 9, 16], those (local) sunspot equilibria are also excluded as limiting behavior of the small random perturbation. Therefore, in those kind of models and in a set of parameter values with full Lebesgue measure, the selection criterion of dynamic equilibria based on the small random perturbation of perfect foresight paths refines the multiplicity of equilibria to following classes of equilibria: determinate cycles or equilibria consistent with an absolutely continuous stationary measure such as a chaotic backward perfect foresight path or a chaotic sunspot equilibrium of the type (2.6).

We will now illustrate the large abundance of strong stochastic stable bpf maps for the OLG model of Section 3.1. We numerically determine values of parameters $(\alpha_1, \alpha_2, l_1^*, l_2^*)$ under which the map $\phi_\lambda$ is a post-critically finite Misiurewicz map, i.e. $\phi_\lambda$ has no periodic attractors and the critical orbit is pre-periodic to a repelling periodic orbit. Note that these are a subset of the set of Misiurewicz maps. To proceed with our numerical experiments, we fix the parameters $l_1^* = 3.51$ and $l_2^* = 0.55$ and work on the two parameter space of relative risk aversion coefficients $(\alpha_1, \alpha_2) \in (0, 1) \times (2, +\infty)$. The results described below are robust with respect to changes in the values of $l_1^*$ and $l_2^*$. We then numerically compute any intersections between the first $N$ iterates of the critical point and the periodic points up to some finite period $M$, excluding all the non-transverse intersections and all the intersections with attracting periodic points. Checking the stability of the periodic points is relevant because in the case where the critical point is pre-periodic to a repelling periodic point, then there are no stable or neutral cycles, since for unimodal maps with negative Schwarzian derivative, these would attract the critical orbit. In Figure 1 it is possible to notice the different dynamical behaviors of the bpf map as the risk aversion parameter $\alpha_2$ increases from 2 to 7.5. For small
values of $\alpha_2$, there exists a unique attracting fixed point of $\phi$. Such attracting fixed point corresponds to a determinate steady state. As $\alpha_2$ increases, attracting periodic points of higher periods are generated by period-doubling bifurcations. Each attracting periodic point corresponds to a determinate cycle of the model. All those $\phi_\lambda$ are ordered bpf maps and lead to invariant measures supported on convex linear combination of Dirac measures supported on the determinate cycle. For large enough values of $\alpha_2$, Misiurewicz maps can be found. See Figure 2 for the distribution of Misiurewicz maps in parameter space $(\alpha_2, \alpha_1) \in (2, 7.5) \times (0.01, 0.29)$. As noted above, the values of parameters under which such maps occur are density points of positive measure sets where stability of small random perturbation of $\phi_\lambda$ can be found. Such parameters are associated with invariant B-R-S measures.

To leave clear the distinction between the behaviors associated with the two possibilities of convergence discussed above, for the parameter values used above and fixing additionally $\alpha_1 = 0.41$ and $\alpha_2 \in \{5.0; 6.5\}$, we plot in Figures 3 and 4 histograms associated with $10^6$ iterations of paths generated by the bpf map $\phi$ for two different set of parameters values and varying sizes of the random perturbation.

The simulations discussed above can be criticized by the use of an excessively high relative risk aversion parameter. However, for the market game model of Section 3.2 with the constant relative risk aversion utility functions (3.21), we present the analogous analysis using more conservative values for that parameter. Specifically, fixing the parameter values $\alpha_1 = 0.5$, $\alpha_2 = 2$, $\omega_1 = 2$, $\omega_2 = 0.39$, $\hat{Q} = 0.1$ and varying $\tilde{Q} \in (0.0, 0.135)$ we can observe in Figure 5 the existence of parameters values determining bpf maps with B-R-S measures for values of $\tilde{Q}$ below 0.098, approximately. For those parameter values we obtain the convergence of the stationary measures of small random perturbation to the absolutely continuous B-R-S measure of the bpf map (and stationary measure of the sunspot equilibria (2.6)). For $\tilde{Q}$ greater than 0.098, attracting cycles of the bpf map arise and therefore we obtain the convergence of the stationary measures of small random perturbations to convex linear combinations of Dirac measures supported on the corresponding determinist cycle. Figures 6 and 7 contain histograms associated with varying maximal sizes for the random perturbations for the parameter values listed above and, respectively, $\hat{Q} = 0.09$ and $\hat{Q} = 0.12$, enabling us to compare once again the chaotic dynamics case against the regular dynamics case associated with an attracting cycle. For the case of unitary relative risk aversion (logarithmic utilities) and values of parameters $\omega_1 = 5.4$, $\omega_2 = 0.5$, $\hat{Q} = 0.2$ and varying $\tilde{Q} \in (0.0, 0.034)$, we observe in Figure 8 that stochastic stability with respect to an absolutely continuous invariant B-R-S measure holds for a wide range of values of $\tilde{Q}$ below 0.255, approximately. The histograms in Figures 9 and 10 are obtained for varying maximal sizes of the random perturbation fixing, additionally, $\hat{Q} = 0.23$ and $\hat{Q} = 0.30$.

4. Conclusions

In this paper we propose a new selection criterion of dynamic equilibria in non-linear one-period forward looking economic models. The criterion is based on the limiting behavior of the stationary process generated by a small random perturbation of the deterministic version of the model. The small random perturbation can be interpreted as small deviations from the perfect foresight dynamics that may arise from coordination failures in expectations, for example.
We apply that selection criterion to models with backward perfect foresight maps which are unimodal maps with negative Schwarzian derivative. For a large class of such models, we obtain that there exists a set of parameter values with full Lebesgue measure where the only equilibria that are selected under this refinement criterion are either determinate cycles (including deterministic steady states) or equilibria consistent with the absolutely continuous invariant measure of the backward perfect foresight map, namely, chaotic backward perfect foresight paths or the stationary measure of the chaotic sunspot equilibrium presented in [1]. Neither indeterminate cycles nor other types of sunspot equilibria are selected by the proposed criterion.

To illustrate the large set of parameter values where those results are valid, we applied numerical simulations to two models in economic dynamics: the OLG model with fiat money and the Shapley-Shubick market game model. For both models we derived the equations defining the intertemporal equilibrium dynamics and showed the set of parameter values exhibiting convergence to a Bowen-Ruelle-Sinai measure, as well as parameter values exhibiting convergence to an atomic measure with support on an attracting cycle. Therefore, the proposed analysis aims at assessing the robustness of those equilibria, namely the global sunspot equilibrium of Araujo and Maldonado [1] or determinate cycles, with respect to the criterion of small random perturbation of perfect foresight equilibrium.

Acknowledgments

We thank the financial support of LIAAD–INESC TEC through program PEst, USP-UP project, Faculty of Sciences, University of Porto, Calouste Gulbenkian Foundation, FEDER and COMPETE Programmes, and Fundação para a Ciência e a Tecnologia (FCT) through the Project “Dynamics and Applications” (PTDC / MAT / 121107 / 2010). M. Choubar’s research was supported by FCT - Fundação para a Ciência e Tecnologia grant with reference SFRH / BD / 51173 / 2010. Wilfredo Maldonado is partially supported by the CNPq of Brazil with the grants 303420/2012-0 and 470923/2012-1.
Appendix A. Sufficient conditions for the strong stochastic stability of a large class of unimodal maps

Below we state a pair of hypotheses which are sufficient to guarantee not only the existence, but also the stochastic stability, of the Araujo and Maldonado [1] chaotic sunspot equilibrium.

Hypothesis A: The backward perfect foresight map $\phi \in F$ satisfies the following conditions:

(A1) Subexponential recurrence: $|\phi^k(x^*) - x^*| \geq e^{-\alpha k}$ for all $k \geq H_0$, 
(A2) Collet–Eckmann condition: $|D\phi^k(\phi(x^*))| \geq e^{\gamma k}$ for all $k \geq H_0$, 
(A3) Visiting property: $\phi$ is topologically mixing in the dynamical interval $X = [\phi^2(x^*), \phi(x^*)]$. 

where $H_0 \geq 1$, $\gamma > 0$ and $0 < \alpha < \gamma/4$ are fixed constants.

Under Hypothesis A, the bpf map $\phi$ restricted to its dynamical interval $X \subset \mathbb{R}^+$, $\phi: X \to X$, has a Bowen-Ruelle-Sinai invariant probability measure, namely, there exists an absolutely continuous measure $\mu$ with support in $X$ such that $\mu(\phi^{-A}) = \mu(A)$ for any Borel set $A \subset X$ and there exists a subset $C \subset X$ with positive Lebesgue measure such that for all $x \in C$ the sequence of measures

$$\mu_N = \frac{1}{N} \sum_{k=0}^{N-1} \delta_{\phi^k(x)}$$

converges to $\mu$ in the weak topology. The unimodal shape of $\phi$ and the existence of an invariant and absolutely continuous measure associated to it are the conditions for the existence of a chaotic sunspot equilibrium, as shown by Araujo and Maldonado [1]. The invariant measure of the Markovian process proposed by them is $\mu$. It is also worth remarking that such measure is ergodic, describing the typical asymptotic behavior of $\phi$ in the sense that $\mu_N \to \mu$ in the weak topology for Lebesgue almost every $x \in X$.

The following hypothesis describes the class of random perturbations under consideration herein.

Hypothesis B: The probability distribution of each term in the process $\{\tilde{\epsilon}_t\}_{t \geq 1}$ has a probability density function $\theta_\epsilon : \mathbb{R} \to \mathbb{R}^+_0$, such that

(i) $\text{supp}(\theta_\epsilon) \subset \Omega_\epsilon = [-\epsilon, \epsilon]$, 
(ii) $\int_{\Omega_\epsilon} \theta_\epsilon(y) \, dy = 1$, 
(iii) $M = \sup_{\epsilon > 0}(\epsilon \sup |\theta_\epsilon|) < \infty$, 
(iv) $J_\epsilon := \{t | \theta_\epsilon(t) > 0\}$ is an interval containing 0 and $\eta_\epsilon := \log(\theta_\epsilon|_{J_\epsilon})$ is a concave function.

Hypotheses (A) and (B) are sufficient to guarantee strong stochastic stability of unimodal maps, as proved by Baladi and Viana [6].

Theorem A.1 (Baladi-Viana). Assume that Hypotheses (A) and (B) hold for a map $\phi \in F$. Then $\phi$ is strongly stochastically stable.

A quick remark regarding the well-posedness of the random dynamical system obtained by perturbing the bpf map $\phi$ by a random noise satisfying hypothesis B is in order. When a stochastic perturbation is introduced, the orbits of such stochastic dynamical system may leave the dynamical interval $X$ associated with
the bpf map $\phi$ with positive probability. However, since $\phi^2(x^*) > 0$ there exists $\epsilon_0 > 0$ such that for every probability density function $\theta_\epsilon$ satisfying hypothesis B, where $\epsilon < \epsilon_0$, there exists a compact interval

$$X_\epsilon = [\phi(\phi(x^*) + \epsilon) - \epsilon, \phi(x^*) + \epsilon] \subset \mathbb{R}^+$$

such that $X \subset X_\epsilon$ and, moreover, $X_\epsilon$ is invariant under both the deterministic and stochastic dynamics associated to $\phi$. Since we will study the limit where the maximum size of the random perturbation $\epsilon \to 0$, we can take the size of the random perturbation to be strictly smaller than $\epsilon_0 > 0$, thus yielding well defined random dynamics in the compact interval $X_\epsilon$.

**Appendix B. Proofs of the main results**

We present below the proofs of the main results presented in this paper.

**Proof of Theorem 2.5.** The result relies heavily on the topological characterization of unimodal maps discussed above and Avila and Moreira dichotomy for smooth unimodal maps [3]. Namely, for a topologically generic $k$-parameter family of bpf maps in $\mathcal{F}$, an open and dense set of parameters correspond to hyperbolic maps. These are stochastically stable and, moreover, the stationary measure associated with a random perturbation of the corresponding perfect foresight path will converge to an atomic measure with support on the (unique) attracting cycle of such hyperbolic map, which corresponds to a determinate equilibrium. By Avila and Moreira dichotomy, for almost every other parameter the bpf map satisfies the subexponential recurrence condition and the Collet-Eckmann condition (see Appendix A), which ensure the existence of an absolutely continuous stationary measure and strong stochastic stability. □

**Proof of Proposition 2.6.** We prove item (i) first. Let $\epsilon > 0$ be small enough so that $\nu^\epsilon$ is the unique invariant and ergodic probability measure associated to $P_\epsilon(\cdot, \cdot)$ (namely, satisfying (2.4)). Furthermore, recall that for $\epsilon > 0$ small enough, $\nu^\epsilon$ is absolutely continuous with respect to the Lebesgue measure. By the definition of the measure $\nu^\epsilon_N$ given in (2.5) and ergodicity of $\nu^\epsilon$, we have that $\nu^\epsilon_N \to \nu^\epsilon$ in the weak$^*$-topology (see [39]). Under Hypothesis A and B, Baladi and Viana Theorem [6] ensures that $\nu^\epsilon \to \mu$ in the weak$^*$-topology. Therefore, combining the two statements above, we obtain that $\nu^\epsilon_N \to \mu$ in the weak$^*$-topology when $N \to \infty$ and $\epsilon \to 0$.

The proof of item (ii) is similar. By ergodicity of $\nu^\epsilon$, we have that $\rho_N^\epsilon, \mu \to \rho^\epsilon$ in $L^1(dx)$ when $N \to \infty$ and the size of largest element of the partition goes to zero (see [39]). The result then follows by noting that Baladi and Viana Theorem [6] also ensures that $\rho^\epsilon \to \rho$ in $L^1(dx)$ as $\epsilon \to 0$. □

**Proof of Theorem 3.3.** For the families of utility functions with constant relative risk aversion introduced in Sections 3.1 and 3.2 and the choice of parameters discussed in items a) and b) above, we have that for every $\lambda \in \Lambda$ the following hold:

- i) $\phi_\lambda$ is a $C^3$ unimodal map;
- ii) $\phi_\lambda$ has a (positive) non-degenerate critical point $\bar{x}(\lambda)$;
- iii) $\phi_\lambda$ has a repelling fixed point at zero.

For one-parameter families of maps satisfying the conditions i), ii) and iii) above and having a Misiurewicz parameter $\lambda_* \in \Lambda$ with generic unfolding, there exists a
positive measure set $A$ in the space of parameters with $\lambda^*$ as a density point and such that conditions (A1) and (A2) from hypothesis A hold for every $\lambda \in A$ (see [19] for further details). As a consequence, for every $\lambda \in A$, we have that

1) $\phi_\lambda$ admits an absolutely continuous invariant measure $\mu_\lambda$, with a $L^p$ density for any $p < 2$;
2) $\mu_\lambda$ is a SBR measure;
3) $\phi_\lambda$ has positive Lyapunov exponent almost everywhere.

Hence, the strong stochastic stability of $\phi_\lambda$ follows from Baladi and Viana Theorem in [6] by observing that condition (A3) holds for unimodal maps with negative Schwarzian derivative and an absolutely continuous invariant measure.

\begin{thebibliography}{19}

\end{thebibliography}
Figure 1. Plot in the $(\alpha_2, x)$ plane of the first 100 iterates of the critical point (in blue) and periodic points (up to period 8) of the bpf map $\phi$ of Section 3.1. We plot the stable periodic points in green and the unstable ones in red. Figure 1a contains only the fixed point of $\phi$, Figure 1b contains the fixed point and the period 2 orbit, Figure 1c contains periodic points of periods 1, 2 and 4, and finally, Figure 1d contains all periodic points whose period divides 8. The remaining parameters are fixed and equal to $l_1^* = 3.51$, $l_2^* = 0.55$ and $\alpha_1 = 0.41$. 
Figure 2. The distribution of Misiurewicz parameters for the family of bpf maps with \((\alpha_2, \alpha_1) \in (2, 7.5) \times (0.01, 0.99)\) for fixed \(l_1^* = 3.51\) and \(l_2^* = 0.55\). These are obtained by considering intersections of the first 100 iterates of the critical point with unstable periodic points of periods 1, 2, 4 and 8.
Figure 3. The bpf $\phi$ and the approximate densities associated with its stationary measure for the overlapping generations model of Section 3.1 and values of parameters $t_1^* = 3.51$, $t_2^* = 0.55$, $\alpha_1 = 0.41$ and $\alpha_2 = 6.5$ corresponding to deterministic bpf dynamics that are chaotic on a finite union of intervals. Fig. (3b) does not contain any randomness, while the maximum size of the random perturbation is $\epsilon = 0.001$ in Fig. (3c) and $\epsilon = 0.00804199$ in Fig. (3d).
Figure 4. The bpf $\phi$ and the approximate densities associated with its stationary measure for the overlapping generations model of Section 3.1 and values of parameters $l_1^* = 3.51$, $l_2^* = 0.55$, $\alpha_1 = 0.41$ and $\alpha_2 = 5.0$ corresponding to deterministic bpf dynamics exhibiting an attracting cycle. Fig. (4b) does not contain any randomness, while the maximum size of the random perturbation is $\epsilon = 0.05$ in Fig. (4c) and $\epsilon = 0.09160628$ in Fig. (4d).
Figure 5. Plot in the $(\tilde{Q}, x)$ plane of the first 100 iterates of the critical point (in blue) and periodic points (up to period 8) of the bpf map $\phi$ of Section 3.2 for the constant relative risk aversion utility functions (3.21). We plot the stable periodic points in green and the unstable ones in red. Figure 5a contains only the fixed point of $\phi$, Figure 5b contains the fixed point and the period 2 orbit, Figure 5c contains periodic points of periods 1, 2 and 4, and finally, Figure 5d contains all periodic points whose period divides 8. The remaining parameters are fixed and equal to $\alpha_1 = 0.5$, $\alpha_2 = 2$, $\omega_1 = 2$, $\omega_2 = 0.39$ and $\tilde{Q} = 0.1$. 
Figure 6. The bpf $\phi$ and the approximate densities associated with its stationary measure for the market game example of Section 3.2 with the constant relative risk aversion utility functions (3.21) and values of parameters $\alpha_1 = 0.5$, $\alpha_2 = 2$, $\omega_1 = 2$, $\omega_2 = 0.39$, $\hat{Q} = 0.1$ and $\tilde{Q} = 0.09$ corresponding to deterministic bpf dynamics that are chaotic on a finite union of intervals. Fig. (6b) does not contain any randomness, while the maximum size of the random perturbation is $\epsilon = 0.0005$ in Fig. (6c) and $\epsilon = 0.00148461345$ in Fig. (6d).
Figure 7. The bpf $\phi$ and the approximate densities associated with its stationary measure for the market game example of Section 3.2 with the constant relative risk aversion utility functions (3.21) and values of parameters $\alpha_1 = 0.5$, $\alpha_2 = 2$, $\omega_1 = 2$, $\omega_2 = 0.39$, $\hat{Q} = 0.1$ and $\tilde{Q} = 0.12$ corresponding to deterministic bpf dynamics exhibiting an attracting cycle. Fig. (7b) does not contain any randomness, while the maximum size of the random perturbation is $\epsilon = 0.005$ in Fig. (7c) and $\epsilon = 0.014360401$ in Fig. (7d).
Figure 8. Plot in the ($\tilde{Q}, x$) plane of the first 100 iterates of the critical point (in blue) and periodic points (up to period 8) of the bpf map $\phi$ of Section 3.2 for the logarithmic utility function (3.22). We plot the stable periodic points in green and the unstable ones in red. Figure 8a contains only the fixed point of $\phi$, Figure 8b contains the fixed point and the period 2 orbit, Figure 8c contains periodic points of periods 1, 2 and 4, and finally, Figure 8d contains all periodic points whose period divides 8. The remaining parameters are fixed and equal to $\omega_1 = 5.4$, $\omega_2 = 0.5$ and $\tilde{Q} = 0.2.$
Figure 9. The bpf $\phi$ and the approximate densities associated with its stationary measure for the market game example of Section 3.2 with the logarithmic utility function (3.22) and values of parameters $\omega_1 = 5.4$, $\omega_2 = 0.5$, $\hat{Q} = 0.2$ and $\bar{Q} = 0.23$ corresponding to deterministic bpf dynamics that are chaotic on a finite union of intervals. Fig. (9b) does not contain any randomness, while the maximum size of the random perturbation is $\epsilon = 0.0005$ in Fig. (9c) and $\epsilon = 0.0017744586$ in Fig. (9d).
Figure 10. The bpf $\phi$ and the approximate densities associated with its stationary measure for the market game example of Section 3.2 with the logarithmic utility function (3.22) and values of parameters $\omega_1 = 5.4$, $\omega_2 = 0.5$, $\hat{Q} = 0.2$ and $\tilde{Q} = 0.30$ corresponding to deterministic bpf dynamics that are chaotic on a finite union of intervals. Fig. (10b) does not contain any randomness, while the maximum size of the random perturbation is $\epsilon = 0.015$ in Fig. (10c) and $\epsilon = 0.0332824589$ in Fig. (10d).